Math 275D Lecture 17 Notes

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1 Strengthening Itô's Formula, and Generalizing the Itô Integral

1.1 Strengthened Itô's formula

Here is the stronger version of Itô's formula.

Theorem 1.1 (Itô's formula)**.** *Let $f \in C^2(\mathbb{R})$ with $\|f'\|_\infty, \|f''\|_\infty \leq M$. Then*

$$f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_t + \frac{1}{2} \int_0^t f''(B_s)\, ds.$$

We want to show that

$$\sum_{\{t_k\}} f'(B_{t_k}) B(\Delta_k) \to \int f(B_s)\, dB_s$$

$$\sum_{\{t_k\}} f''(B_{t_k}) B(\Delta_k)^2 \to \int f(B_s)\, dB_s$$

where $B(\Delta_k) := B(t_{k+1}) - B(t_k)$.

For the first statement, we need to show that $\sum_{\{t_k\}} f'(B_{t_k}) B(\Delta_k)$ has a limit when the mesh size of $\{t_k\} \to 0$. What does this convergence mean? This is not just a sequence; it is a net.

Definition 1.1. A **net** is a partially ordered collection $\{X_\alpha\}_{\alpha \in I}$ such that for any α, β, there exists a γ such that $\gamma > \alpha$ and $\gamma > \beta$.

For partitions, we have a net: for $T = \{t_k\}$ and $\hat{T} = \{\hat{t} - k\}$, $T \leq \hat{T} \iff \{t_k\} \subseteq \{\hat{t}_k\}$.

Proof. We want to show that this net is Cauchy. Write $B(\Delta_k) = \sum_{\hat{\ell}} B(\Delta_{\hat{\ell}})$, where the $\hat{t}_\ell \in [t_k, t_{k+1}]$. We have

$$\mathbb{E}\left[\sum_k f'(B_k) B(\Delta_k) - \sum_{\hat{k}} f'(B_{\hat{k}}) B(\Delta_{\hat{k}})\right]^2 = \mathbb{E}\left[\sum_k \sum_{\hat{t}_\ell \in [t_k, t_{k+1}]} [f'(B_k) - f'(B_{\hat{\ell}})] B(\Delta_{\hat{\ell}})\right]^2$$

$B(\Delta_{\hat{\ell}})$ is independent of random variables measurable with respect to $\mathcal{F}_{\hat{t}_\ell}$, so a lot of the terms cancel (because they have zero expectation).

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_k \sum_{\hat{t}_\ell \in [t_k, t_{k+1}]} [f'(B_k) - f'(B_{\hat{\ell}})]^2 B(\Delta_{\hat{\ell}})^2\right] \\
&= \mathbb{E}\left[\sum_k \sum_{\hat{t}_\ell \in [t_k, t_{k+1}]} |f'(B_k) - f'(B_{\hat{\ell}})|^2 \Delta_{\hat{\ell}}\right] \\
&\le \sup_{\substack{s,t \in [0,T] \\ |s-t| \le \max t_{k+1} - t_k}} \mathbb{E}[|f'(B_s) - f'(B_t)|]^2 \cdot T
\end{aligned}
$$

This goes to 0 if $\max_k(t_{k+1} - t_k) \to 0$. So this is a Cauchy net, and thus it has a limit.

For the second statement we want to prove, compare

$$\sum_k f''(B_{t_k}) B^2(\Delta_k), \sum_k f''(B_{t_k})\Delta_k.$$

The right term has the limit $\int f''(B_s)\, ds$. Subtracting these two gives

$$\sum_k f''(B_{t_k})[B(\Delta_k)^2 - \Delta_k]$$

Then $f''(B_{t_k})$ is $\mathcal{F}_{t_k}$-measurable, and $B(\Delta_k)^2 - \Delta_k$ is independent of $\mathcal{F}_{t_k}$-measurable random variables and has zero expectation.

In general, for these kinds of sums, we have

$$\mathbb{E}\left[\sum_k g_k h_k\right] = \sum_{k,k'} \mathbb{E}[g_k g_{k'} h_k h_{k'}]$$

If $k' > k$, then $g_k g_{k'} h_{k'} \in \mathcal{F}_{t_{k'}}$. But $h_k \perp \mathcal{F}_{t'_k}$.

$$
\begin{aligned}
&= \sum_k \mathbb{E}[g_k^2 h_k^2] \\
&\le M^2\, \mathbb{E}\left[\sum_k h_k^2\right] \\
&\le M^2 \cdot T \cdot C \cdot \max_k(\Delta_k).
\end{aligned}
$$

So we get that $\sum_k f''(B_{t_k})[B(\Delta_k)^2 - \Delta_k] \to 0$ in L^2, which implies convergence in probability. $\square$

1.2 The Itô integral for more general functions

We have defined $\int f(B_s)\,dB_s$ when $f \in C^2$. What if f depends on the entire path of Bronian motion until time t? The general case is

$$\int_0^T f(\omega, s)\,dB_s, \qquad \omega \in \Omega_B.$$

Example 1.1. Let $f(\omega, t) = \max_{s \le t} |B_s(\omega)|$. Then we are looking at

$$\int_0^T \max_{s \le t} |B_s|\,dB_t.$$

What kind of function should f be?

1. If Brownian motion has the measure space $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$, then f should be $\mathcal{F}_B \otimes \mathcal{T}$-measurable.
2. For fixed t, $f(\omega, t) \in \mathcal{F}_t$.
3. $f \in L^2(\Omega_B \times [0, T])$: that is, $\mathbb{E}[\int f^2(\omega, t)\,dt] < \infty$.

We let $\mathcal{H}$ be the collection of f satisfying these 3 properties. We start from

$$f(\omega, t) = a(\omega) \cdot \mathbb{1}_{[t_1, t_2]}(t).$$

What property should $a(\omega)$ satisfy to satisfy property (2) above? We want $a \in \mathcal{F}_{t_1}$.